A Remark on K-irreducible Operators

Bit-Shun Tam
Tamkang University

TAMKANG JOURNAL

淡 江 學 報

（ 抽 印 本 ）

A Remark on K-Irreducible Operators

譚必信

Bit-Shun Tam

數學系專任教授

No.29

第 二 十 九 期

中 華 民 國 七 十 九 年 八 月 出 版

Tamkang Journal, no. 29: 465-467 (1990)

A Remark on K-Irreducible Operators

譚 必 信
Bit-Shun Tam
數學系專任敎授

Abstract

Let K be a proper cone in a finite-dimensional real vector space V. Denote by $\pi(K)$ the set of all linear operators on V that preserve K. For any $A \in \pi(K)$, denote by S_A the set $\{x \in \partial K: A^i x \in \partial K \text{ for } i = 1,2,\cdots\}$. In this paper we prove the following result: Let A be a K-irreducible operator. Then $S_A = \partial K$ if and only if $AK = K$. Our result resolves an open problem posed by Barker.

1. Introduction

Let V be a finite-dimensional real vector space, equipped with the usual topolgy (that is, the unique topolgy which turns V into a Hausdorff topological vector space). A subset K of V is called a *proper cone* of V if the following conditions are satisfied: (1) $\alpha x + \beta y \in K$ for any $x, y \in K, \alpha,\beta \geqslant 0$, (2) $K \cap (-K) = \{O\}$, (3) K is (topologically) closed, and (4) $K - K = V$; or equivalently, int K (the interior of K) is nonempty. Hereafter, we always use K to denote a proper cone in V.

By a *face* F of K is meant a subset of K which satisfies conditions (1),(2),(3) above and the following condition: if $x, y \in K$ such that $x + y \in F$ then $x, y \in F$. For basic properties of faces of a cone, see Barker [1].

Denote by $\pi(K)$ the set of all linear operators A on V such that $AK \subset K$. If $A \in \pi(K)$ and A leaves invariant no face of K except $\{O\}$ and K itself, A is said to be *K-irreducible.* For other equivalent definitions on K-irreducible operators, see Vandergraft [3,Theorem 4.1 and Lemma 4.2]. A linear operator $A \in \pi(K)$ is said to be *K-primitive,* if there exists a positive integer m such that $A^m x \in \text{int } K$ for all nonzero vectors $x \in K$. (Also see Barker [2, Definition 3] for an alternative equivalent definition.)

It is well-known that every K-primitive operator is K-irreducible. This fact is also clear, by the following result (Barker [2, Theorem 1]): for any $A \in \pi(K)$, A is K-primitive iff A leaves no subset of ∂K (the boundary of K) other than $\{O\}$ invariant.

For any $A \in \pi(K)$, we denote by S_A the set $\{x \in \partial K: A^i x \in \partial K$ for all positive integers $i\}$.

A K-irreducible operator which is not K-primitive is said to be *K-imprimitive.* As pointed out by Barker [2, Prop. 3], when A is K-imprimitive, there is a maximal nonzero invariant subset S in ∂K. It is not difficult to show that this maximal subset is in fact the set S_A. (This fact, however, was not mentioned explicitly by Barker [2].)

In [2, Theorem 4] Barker proved that if dim V = 3, A is K-imprimitive and S_A is arcwise connected, then $S_A = \partial K$. His proof depends on the topology of 3-space and cannot be carried over to higher dimensional spaces. In Theorem 5 of the same paper Barker also proved that if A is K-irreducible and nonsingular, then $S_A = \partial K$ iff $A^{-1} \in \pi(K)$. But he left open the question of determining when $S_A = \partial K$ for general K-irreducible operators A. In this paper we shall answer this question by showing that, without the nonsingularity assumption, Theorem 5 in Barker's paper still holds.

We also take this opportunity to point out a related question posed by Barker [2]. If A is K-irreducible, then by definition for each $x \in \partial K \backslash S_A$ there is a positive integer k such that $A^k x \in \text{int } K$. Theorem 3 in [2] shows that if K is polyhedral, then the k may be chosen independently of x. Barker asked the question of whether k can be taken independently of x for arbitrary proper cones K. This question is still open, and appears to be more difficult.

2. Main Results

We denote by Aut(K) the set $\{A \in \pi(K): A^{-1}$ exists and belongs to $\pi(K)\}$. It is clear that $A \in \text{Aut}(K)$ iff $AK = K$. Under the composition of maps, Aut(K) forms a group, known as the *automorphism group* of K.

LEMMA. *Let $A \in \pi(K)$ be nonsingular. Then $S_A = \partial K$ if and only if $A \in Aut(K)$.*

Proof. "If" part. Then A is a homeomorphism which takes K onto itself. Hence $A(\partial K) = \partial K$ (and $A(\text{int } K) = \text{int } K$). Thus for every vector $x \in \partial K$, $A^i x \in \partial K$ for all positive integers i. In other words, $S_A = \partial K$.

"Only if" part. First, we show that the nonsingularity assumption on A implies that if $x \in \text{int } K$, then $Ax \in \text{int } K$. Indeed, assume the contrary that $Ax \in \partial K$. Denote by $\leqslant_K$ the partial ordering on V induced by K; that is, $x \leqslant_K y$ iff $y-x \in K$. Then for any vector $y \in K$, since $x \in \text{int } K$, we have, $y \leqslant_K \alpha x$ for some $\alpha > 0$. As $A \in \pi(K)$, it follows that $Ay \leqslant_K \alpha Ax$. This shows that A takes K into $\phi(Ax)$ (the face of K generated by Ax), and hence AK lies in the boundary of K. This contradicts the nonsingularity assumption on A.

To prove that $A \in Aut(K)$, it suffices to show that $AK \supset K$. Assume the contrary that there exists a vector $y \in K$ such that its (unique) pre-image under A, say x, is outside K. Choose some vector $u \in \text{int } K$. Then there exists λ, $0 < \lambda < 1$, such that $\lambda u + (1-\lambda)x \in \partial K$. But $A(\lambda u + (1-\lambda)x) = \lambda Au + (1-\lambda)y \in \text{int } K$, as $Au \in \text{int } K$ according to our beginning remark. Thus the vector $\lambda u + (1-\lambda)x$ does not belong to S_A, and so $S_A \neq \partial K$. The proof is complete.

THEOREM. *Let* $A \in \pi(K)$ *be K-irreducible. Then* $S_A = \partial K$ *if and only if* $A \in Aut(K)$.

Proof. By the above lemma, the "if" part is obvious. Also, to establish the "only if" part it suffices to show that when A is K-irreducible and $S_A = \partial K$, A must be non-singular. Indeed, assume that the contrary holds. Choose some vector $u \in \text{int } K$. Since A is K-irreducible, necessarily, $Au \in \text{int } K$; otherwise, by the argument given in the proof of the "if" part of the above lemma, A takes K into $\phi(Au)$, and hence A leaves invariant the nontrivial face $\phi(Au)$ of K. Now choose some vector $y \in \eta(A) \setminus K$, where $\eta(A)$ denotes the nullspace of A. (Such a vector clearly exists, since $\eta(A) \neq \{0\}$ and $K \cap (-K) = \{0\}$.) Then $u + \alpha y \in \partial K$ for some $\alpha > 0$. Furthermore, we have, $A(u + \alpha y) = Au \in \text{int } K$. Hence $u + \alpha y \notin S_A$, and so $S_A \neq \partial K$. The proof is complete. ■

Finally, we note that the "only if" part of our theorem no longer holds if we drop the K-irreducibility assumption on A. Indeed, let $A \in \pi(K)$ and suppose that A takes K into a nontrivial face of K. Then clearly $A \notin Aut(K)$ but $S_A = \partial K$.

References

1. G.P. Barker, The lattice of faces of a finite-dimensional cone, *Linear Algebra Appl.* 7: 71-82 (1973).
2. G.P. Barker, On matrices having an invariant cone, *Czechoslovak Math. J.* 22 (97): 49-68 (1972).
3. J.S. Vandergraft, Spectral properties of matrices which have invariant cones, *SIAM J. Appl. Math.* 16: 1208-1222 (1968).